MALTHUSIAN MODEL OF GROWTH

23. POPULATION

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WORKER'S OPTIMAL CONSUMPTION

- a worker chooses consumption and number of children
  - to maximize her utility
  - subject to her budget constraint
- hence: a worker chooses c(t)
  - to maximize $n(t)^{\beta} \times c(t)^{1-\beta}$
  - where $n(t) = [y(t) - c(t)] \;/\; p$

WORKER'S OPTIMAL CONSUMPTION

- a worker chooses c(t) to maximize $[y(t) - c(t)]^{\beta} \times c(t)^{1-\beta}$
  - because $n(t) = [y(t) - c(t)] / p$
  - we omit the term p^{β}, which does not change the maximization
- the optimal consumption is $c(t) = (1-\beta) \times y(t)$
  - the worker keeps a fraction $1-\beta$ of the food produced to herself
  - the worker gives a fraction β of the food produced to her children
- the worker consumes less when
  - she produces less food
  - she values children more (because then she has more children)

DETAILS OF MAXIMIZATION

- we find c to maximize the function $f(c) = [y - c]^{\beta} \times c^{1-\beta}$

- the function f(c) is maximized when the function $g(c) = \ln(f(c)) = \beta \times \ln(y - c) + (1 - \beta) \times \ln(c)$ is maximized

- at the maximum, $g'(c) = - \beta / (y - c) + (1 - \beta) / c = 0$

- $\beta / (y - c) = (1 - \beta) / c$

- $\beta \times c = (1 - \beta) \times (y - c)$

- $\beta \times c + (1 - \beta) \times c = (1 - \beta) \times y$

- hence: $c = (1 - \beta) \times y$

WORKER'S OPTIMAL NUMBER OF CHILDREN

- to satisfy the budget constraint, the number of children must be $n(t) = [y(t) - c(t)] / p$
- a worker's optimal consumption is $c(t) = (1 - \beta) \times y(t)$
- so the optimal number of children is $n(t) = \beta\, y(t) / p$
- a worker has more children when
  - she produces more food (high y)
  - she enjoys children more (high β)
  - children eat less (low p)

FERTILITY RATE

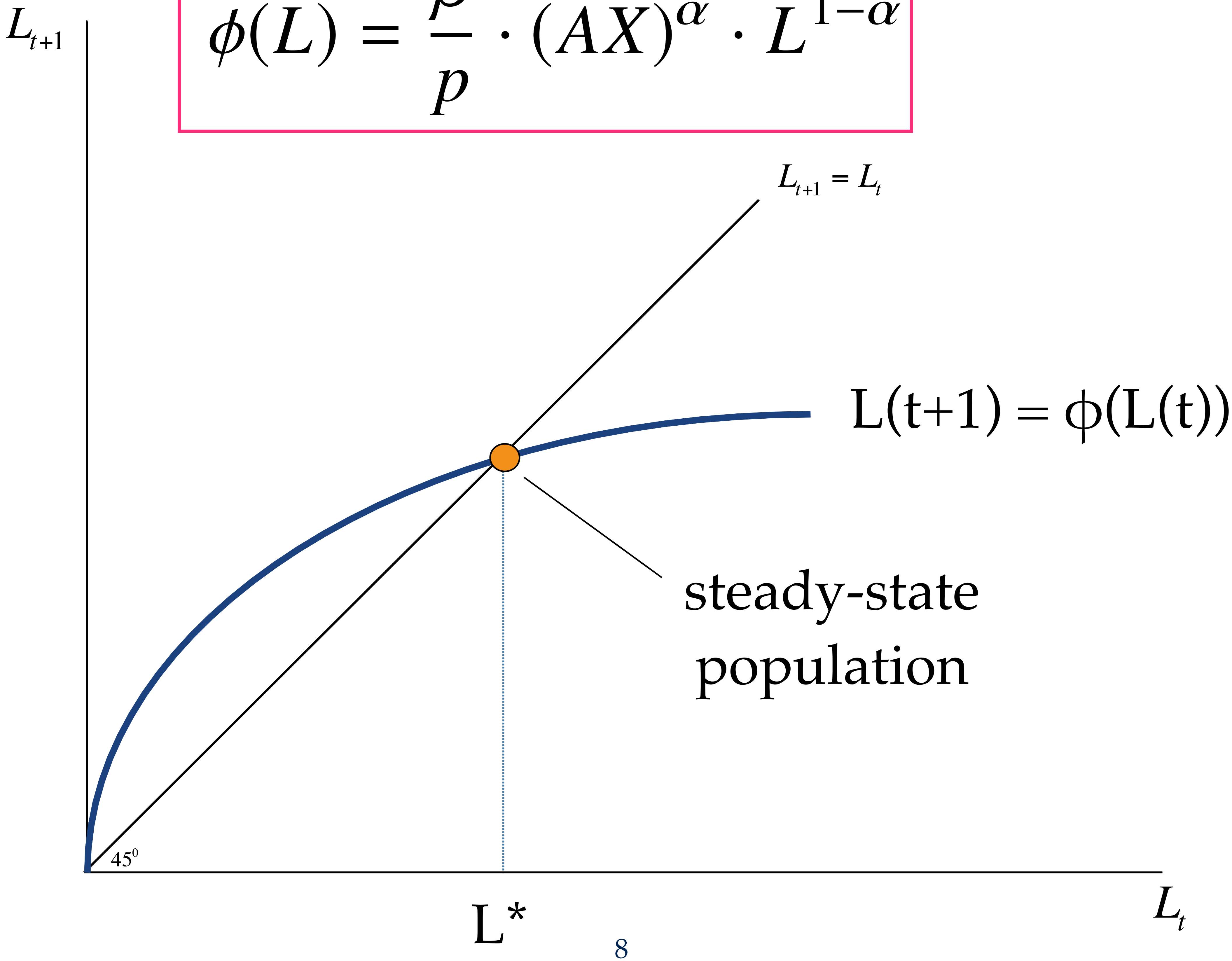
- the fertility rate is the number of children per adult, $n(t)$
  - we saw: $n(t) = \beta \times y(t) \;/\; p$ and $y(t) = [A \;\; X \;/\; L(t)]^{\alpha}$
  - so $n(t) = (\beta / p) \times [A \;\; X \;/\; L(t)]^{\alpha}$
- the working population at t+1 is the children population at t
- this is the working population at time t × the fertility rate at time t
- hence $L(t+1) = n(t) \times L(t) = (\beta / p) \times [A \;\; X \;/\; L(t)]^{\alpha} \times L(t)$
- law of motion of population: $L(t+1) = (\beta / p) \times (AX)^{\alpha} \times L(t)^{1-\alpha}$

STEADY-STATE WORKING POPULATION

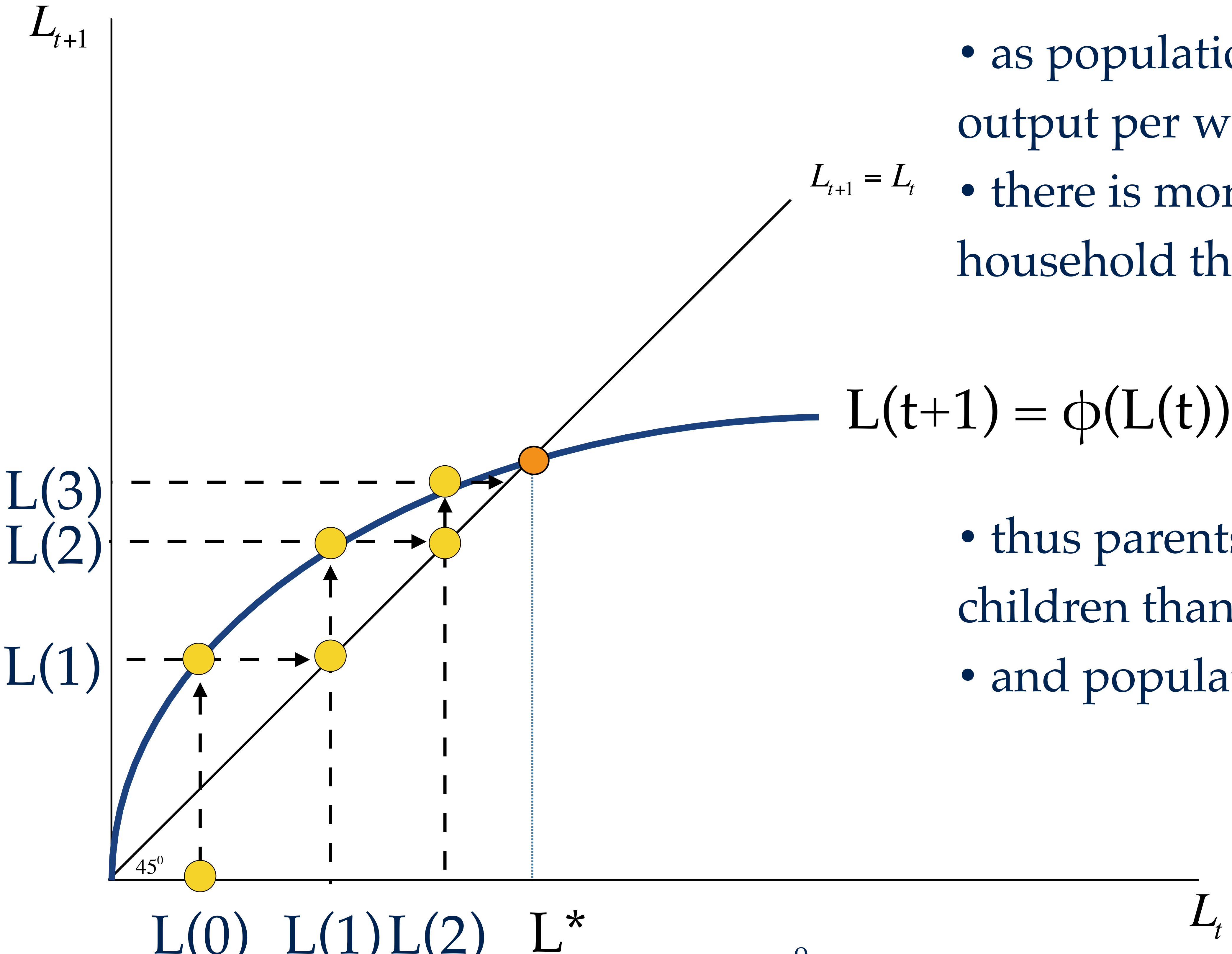
- the population dynamics are given by $L(t+1) = \phi(L(t))$
  - where the function $\phi(L) = (\beta/p) \times (A\,X)^{\alpha} \times L^{1-\alpha}$
- steady-state working population satisfies:
  - $L(t+1) = L(t)$
  - once population is in steady state, it does not change
- so in steady state $L^* = \phi(L^*)$
- $L^* = (\beta/p) \times (A\,X)^{\alpha} \times (L^*)^{1-\alpha}$
- hence in steady state: $L^* = (\beta/p)^{1/\alpha} \times (A\,X)$

STEADY-STATE WORKING POPULATION

$$\phi(L) = \frac{\beta}{p} \cdot (AX)^{\alpha} \cdot L^{1-\alpha}$$



DYNAMICS OF WORKING POPULATION



- as population is below L*, output per worker is above y*
- there is more food per household than in steady state

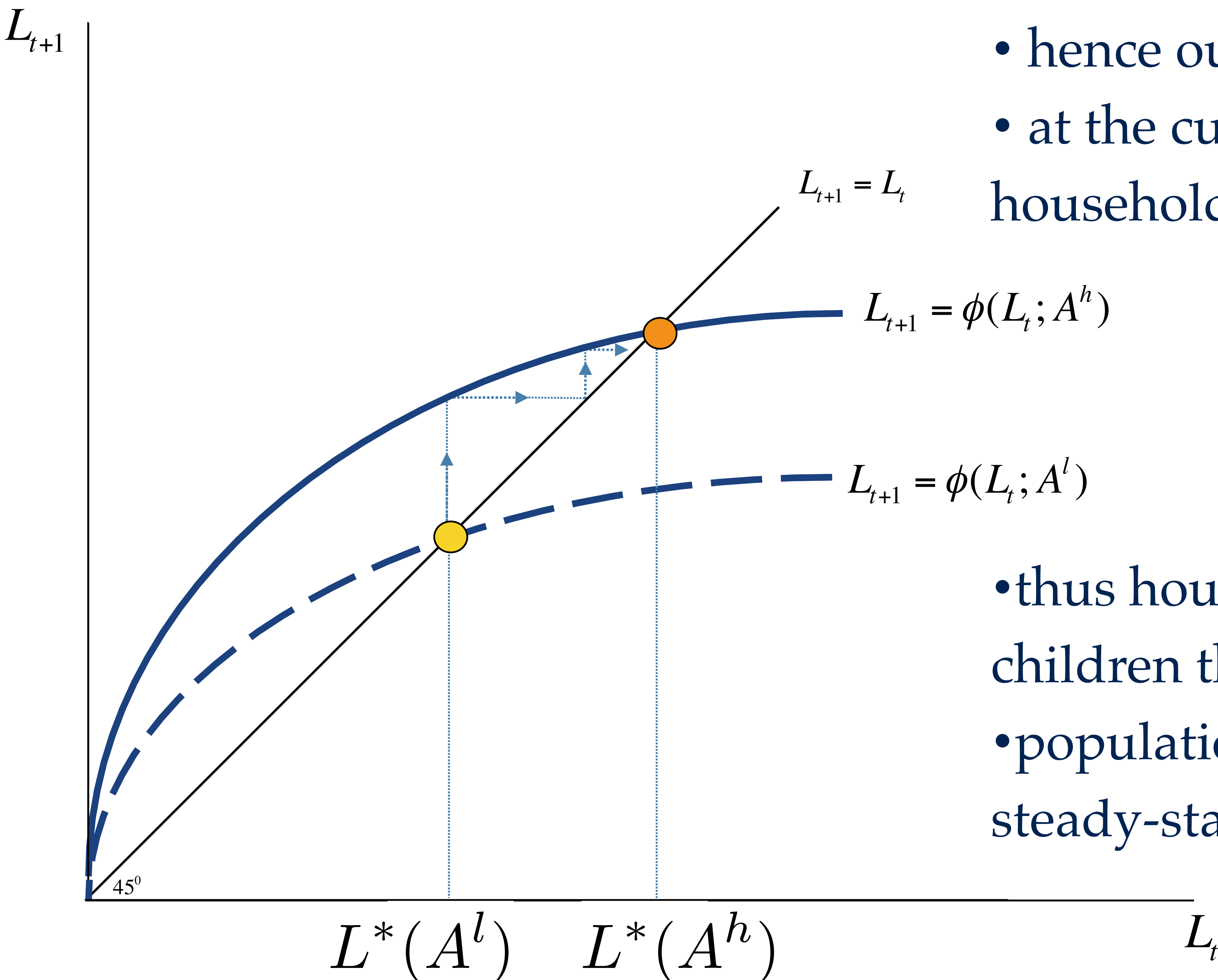
- thus parents have more children than in steady state
- and population is growing

DETERMINANTS OF LONG-RUN POPULATION

- steady-state working population is higher when
  - there is more land (high X)
  - technology is higher (high A)
  - people value children more (high β)
  - children eat less (low p)
- in steady state population is constant so the fertility rate is 1
- in steady state total population is twice the working population: same number of workers and children

POPULATION WITH BETTER TECHNOLOGY

$$\phi(L(t)\,;A) = (\beta/p)\,(A\,X)^{\alpha}\,L(t)^{1-\alpha}$$



- technology improves from A^l to $A^h > A^l$
- hence output increases
- at the current population, households have more food

- thus households have more children than in steady state
- population grows to a higher steady-state value