Household's Problem

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Household's problem. Choose $c(t)$ & $w(t)$ to maximize

$$\int_0^{\infty} e^{-\delta t} \left[\frac{\varepsilon}{\varepsilon - 1} c(t)^{\frac{\varepsilon-1}{\varepsilon}} + \sigma\left(w(t) - \bar{w}(t) \right) \right] dt$$

subject to
$$\dot{w}(t) = r(t)\, w(t) + a\left[1 - u(t)\right] h - \left[1 + \tau(\theta(t))\right] c(t) - T(t)/p(t)$$

+ no Ponzi condition.

Household takes as given $\theta(t)$, $u(t)$, $p(t)$, $T(t)$, $i(t)$, and initial real wealth $w(0)$.

Hamiltonian (current value).

control, state variable

$$\mathcal{H}(t, c(t), w(t)) = \frac{\varepsilon}{\varepsilon-1} c(t)^{\frac{\varepsilon-1}{\varepsilon}} + \sigma\left(w(t) - \bar{w}(t)\right) + \gamma(t)\left[r(t)\, w(t) + \left[1 - u(t)\right] a h - \left[1 + \tau(\theta(t))\right] c(t) - T(t)/p(t) \right]$$

costate variable

Necessary conditions for optimality:

- $\partial \mathcal{H}/\partial c = 0$
- $\partial \mathcal{H}/\partial w = \delta\, \gamma(t) - \dot{\gamma}(t)$

- appropriate transversality condition

$$\lim_{t \to \infty} e^{-\delta t} \gamma(t) w(t) = 0$$

Theorem 7.13 in Acemoglu (2009)

Theorem 7.14 in Acemoglu (2009) $\rightarrow$ any interior solution to necessary conditions is global maximum.

Euler equation:

- $\partial H / \partial c = 0 \Rightarrow \frac{\varepsilon}{\varepsilon - 1} \times \frac{\varepsilon - 1}{\varepsilon} \times c^{-1/\varepsilon} - \gamma(t)\left[1 + \tau(\phi(t))\right] = 0$

$$\Rightarrow \boxed{c(t)^{-1/\varepsilon} = \gamma(t)\left[1 + \tau(\phi(t))\right]}$$

- $\partial H / \partial w = \delta \cdot \gamma(t) - \dot{\gamma}(t) \Rightarrow \sigma'(w(t) - \bar{w}(t)) + \gamma(t) r(t) = \delta \gamma(t) - \dot{\gamma}(t)$

$$\Rightarrow \boxed{\dot{\gamma}(t) = [\delta - r(t)]\gamma(t) - \sigma'(w(t) - \bar{w}(t))}$$

Special case w/ $\tau = 0$, $\sigma' = 0$:

$$\begin{cases} c(t)^{-1/\varepsilon} = \gamma(t) \\ \dot{\gamma} = [\delta - r(t)]\gamma(t) \end{cases}$$

standard Euler equation

$$\begin{cases} \dot{\gamma}/\gamma = \delta - r \\ -1/\varepsilon \; \dot{c}/c = \dot{\gamma}/\gamma \end{cases} \Rightarrow \boxed{\dot{c}/c = \varepsilon(r - \delta)}$$